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2010 J. Phys. A: Math. Theor. 43 062001

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## FAST TRACK COMMUNICATION

# Renormalization group and Lienard systems of differential equations

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Received 26 October 2009

Published 19 January 2010

Online at [stacks.iop.org/JPhysA/43/062001](http://stacks.iop.org/JPhysA/43/062001)**Abstract**

Autonomous Lienard systems, which constitute a huge family of periodic motions, exhibit limit cycle behaviour in certain cases and centres in others. In the literature, the signature for the existence of these two different facets of periodic behaviour has been studied from different geometrical perspectives and not from a general viewpoint. Starting out from general considerations, we show in this work that a certain renormalization scheme is capable of unifying these two different aspects of periodic motion. We show that the renormalization group allows a unified analysis of the limit cycle and centre in a Lienard system of differential equations. While the approach is perturbative, it is possible to make a stronger statement in this regard. Two different classes of Lienard systems have been considered. The analysis provides clear insight into how the frequency gets corrected at different orders of perturbation as one flips the parity of the 'damping' term.

PACS number: 05.10.Cc

The most commonly occurring periodic motions in physical, chemical and biological systems, as well as in engineering applications, belong to second-order autonomous Lienard systems of differential equations. In the literature [1–3] extensive studies exist pertaining to renowned oscillators which go after the names of Duffing, Van der Pol, Raleigh, to mention just a few. In practice, it is often necessary to understand the specific nature of the system under consideration, namely whether its trajectory in phase space succumbs to a limit cycle or revolves around a centre. Various theorems and methods have been developed to address this question in the context of Lienard systems and even beyond. These theorems have elegant geometric proofs [4–6] which are different for different situations. However, a unified way of probing whether a system has a limit cycle or centre is lacking in the literature. Integrability of a class of Lienard and related systems has seen a great deal of progress in the recent years [7–10], but our concern here will be with the issue of limit cycle as opposed to the nonlinear centre.

Studying differential equations from the point of view of renormalization group (RG) symmetries has been an interesting and challenging field of research for some years. Several detailed and authoritative reviews exist in the field [11]. A decade back, a different way of looking at problems of nonlinear dynamics of oscillations has been proposed by Chen *et al* [12] and has been explored by several groups [13]. This involves a *direct* use of perturbation theory and the renormalization group. The advantage lies in the fact that an initial ansatz [4, 14, 15] for the form of the final solution is not required. For detailed pedagogic surveys, we refer the reader to two recent works [16, 17]. This renormalization method opens a new avenue for studying the dynamics of Lienard systems from a general parlance. We begin with a brief explanation of the method itself and then proceed to more general considerations.

We introduce the renormalization method here very briefly through a simple example of a quartic oscillator:

$$\ddot{x} = -\omega^2 x - \lambda x^3 \tag{1}$$

where  $\lambda$  is supposedly small. To solve the equation perturbatively, one starts with the expansion  $x(t) = x_0(t) + \lambda x_1(t) + \dots$  and obtains the zeroth-order (unperturbed) equation as  $\dot{x}_0 + \omega^2 x_0 = 0$  with the harmonic solution  $x_0 = a \cos(\omega t + \theta)$ . Using this solution in the first-order equation  $\dot{x}_1 + \omega^2 x_1 = -x_0^3$ , the solution  $x(t)$  is obtained as

$$x = a \cos(\omega t + \theta) + \lambda \left[ -\frac{3a^3}{16\omega^2} \cos(\omega t + \theta) - \frac{a^3}{32\omega^2} \cos 3(\omega t + \theta) \right] - \frac{3a^3}{8\omega} \cdot \lambda t \cdot \sin(\omega t + \theta). \tag{2}$$

The last term is divergent. In the renormalization scheme, this secular divergence is contained by introducing an *arbitrary* time scale  $\mu$  and writing the divergent term as  $-\frac{3a^3}{8\omega} \cdot \lambda(t - \mu + \mu) \cdot \sin(\omega t + \theta)$ . At this point renormalization constants  $Z_1(\mu)$  and  $Z_2(\mu)$  are introduced as  $a(0) = Z_1(\mu)a(\mu) = a(\mu)(1 + A_1\lambda + \dots)$  and  $\theta(0) = \theta(\mu) + Z_2(\mu) = \theta(\mu) + B_1\lambda + \dots$ . These perturbative expansions of  $Z_1$  and  $Z_2$  are defined in such a way that the  $\mu$  divergence in  $\mu \sin(\omega t + \theta)$  is nullified. Accordingly, we find  $A_1 = 0$  and  $B_1 = \mu \frac{3a^2(\mu)}{8\omega}$  and equation (2) becomes

$$x = a(\mu) \cos(\omega t + \theta(\mu)) + \lambda \left[ -\frac{3a^3(\mu)}{16\omega^2} \cos(\omega t + \theta(\mu)) - \frac{a^3(\mu)}{32\omega^2} \cos 3(\omega t + \theta(\mu)) - (t - \mu) \frac{3a^3(\mu)}{8\omega} \sin(\omega t + \theta(\mu)) \right]. \tag{3}$$

Since  $\mu$  is an arbitrary time scale, the dynamics is independent of  $\mu$  and hence  $\frac{dx}{d\mu} = 0$ . This is the RG equation which, on equating the coefficients of  $\cos(\omega t + \theta(\mu))$  and  $\sin(\omega t + \theta(\mu))$ , yields the amplitude and phase equations, respectively, as

$$\begin{aligned} \frac{da}{d\mu} = 0 &\Rightarrow a \text{ is a constant,} \\ \frac{d\theta}{d\mu} = \frac{3a^2\lambda}{8\omega} &\Rightarrow \theta = \mu \frac{3a^2\lambda}{8\omega}. \end{aligned} \tag{4}$$

The remaining  $\mu$  dependence is removed by setting  $\mu = t$  (because  $\mu$  is arbitrary). Thus,

$$x = a \cos \left( \omega + \frac{3a^2\lambda}{8\omega} \right) t - \lambda \left[ \frac{3a^3}{16\omega^2} \cos(\omega t + \theta) + \frac{a^3}{32\omega^2} \cos 3(\omega t + \theta) \right]. \tag{5}$$

This is the renormalized form of  $x$  to the first order. The important point to note is that if our aim had been to obtain only the amplitude and phase equations (equation (4)), we could have directly written them by merely inspecting equation (2). The recipe is simple. If

the divergent term is of the form  $A_1 \cdot t \cos(\omega t + \theta)$ , where  $A_1$  is a constant, then this term contributes to the *amplitude* equation as  $\frac{da}{d\mu} = A_1$ . On the other hand, if the divergent term is of the form  $A_2 \cdot t \sin(\omega t + \theta)$ , where  $A_2$  is a constant, then this term contributes to the *phase* equation as  $\frac{d\theta}{d\mu} = -\frac{A_2}{a}$ . In equation (2), there was no  $t \cos(\omega t + \theta)$  term and hence we obtained  $\frac{da}{d\mu} = 0$ . It only contained a divergent term of the form  $(-\frac{3a^3}{8\omega})t \sin(\omega t + \theta)$  and hence for  $\frac{d\theta}{d\mu}$  we obtained  $(-1/a)(-\frac{3a^3}{8\omega}\lambda)$ . In this process we note a vital point. This system has a centre-like oscillation and we find that the flow equation gives  $\frac{da}{d\mu} = 0$ . This is intuitively correct since the amplitude is fixed by initial conditions and hence cannot flow. For the limit cycle, on the other hand, we must have a fixed point in the dynamics of  $\frac{da}{d\mu}$ .

We now turn our attention to a general Lienard system exhibiting periodic solution. More precisely, we focus our study on Lienard systems of the *first* kind. To allow the use of perturbation theory we write it in the form

$$\ddot{x} + \epsilon f(x)\dot{x} + x + \alpha g(x) = 0. \tag{6}$$

We make no claim that, mathematically, this is the most general form of Lienard system conceivable. But, this system, of course, is sufficiently general as to encompass a very wide class of periodic motion occurring ubiquitously in various branches of physical, chemical as well as biological sciences. At the end of the paper, we will allude to another class of Lienard system, which we call the Lienard system of the second kind. The form of equation (6) ensures that, in the unperturbed condition, the system executes harmonic motion. The primary requirement for a periodic motion is a trapping potential which is achieved by setting the function  $g(x)$  as odd. Thus, we write

$$g(x) = \sum_{n=1}^{\infty} d_n x^{2n+1} \tag{7}$$

as a series in odd powers of  $x$ .

Our aim is to investigate what happens when  $f(x)$  is an odd function and what happens when it is even. We intend to do this investigation in a *unified* way by writing  $f(x)$  as a sum of two series: one consisting of even powers of  $x$  and the other consisting of odd powers of  $x$ . For example, when we are interested in the consequences of  $f(x)$  being an odd function, we will simply put off the even coefficients from our general results and work only with the odd coefficients. Thus,

$$f(x) = \sum_{n=0}^{\infty} [b_n x^{2n} + c_n x^{2n+1}]. \tag{8}$$

We expand  $x$  in  $\epsilon$  and  $\alpha$  as

$$x(t) = x_0(t) + \epsilon x_{1\epsilon}(t) + \alpha x_{1\alpha}(t) + \epsilon^2 x_{2\epsilon}(t) + \alpha^2 x_{2\alpha}(t) + \epsilon\alpha x_{\epsilon\alpha}(t) + \dots \tag{9}$$

It should be clear from the notation that  $(x_{1\epsilon}, x_{1\alpha})$  are the first-order terms and  $(x_{2\epsilon}, x_{2\alpha}, x_{\epsilon\alpha})$  are the second-order terms. To linear order in  $\epsilon$  and  $\alpha$ ,

$$\ddot{x}_0 + x_0 = 0 \tag{10}$$

$$x'_{1\epsilon} + x_{1\epsilon} = -\dot{x}_0 f(x_0) \tag{11}$$

$$x''_{1\alpha} + x_{1\alpha} = -g(x_0). \tag{12}$$

The zeroth-order equation given by equation (10) immediately yields

$$x_0 = A \cos t + B \sin t = a \cos(t + \theta). \tag{13}$$

We now turn to the first-order system. The right-hand side of equation (11) can be written as

$$\begin{aligned}
 -\dot{x}_0 f(x_0) &= -\dot{x}_0 \sum_{n=0}^{\infty} (b_n x_0^{2n} + c_n x_0^{2n+1}) \\
 &= -\frac{d}{dt} \sum_{n=0}^{\infty} \left[ \frac{b_n}{2n+1} x_0^{2n+1} + \frac{c_n}{2n+2} x_0^{2n+2} \right]
 \end{aligned} \tag{14}$$

and for the right-hand side of equation (12) we have from equation (7)

$$g(x_0) = \sum_{n=1}^{\infty} d_n x_0^{2n+1}. \tag{15}$$

For a given  $n$ , with  $x_0 = a \cos(t + \theta)$  we have

$$\begin{aligned}
 x_0^{2n+1} &= a^{2n+1} \cos^{2n+1}(t + \theta) \\
 &= a^{2n+1} \sum_{k=0}^n \frac{1}{2^{2n}} C_k^{2n+1} \cos(2n - 2k + 1)(t + \theta),
 \end{aligned} \tag{16}$$

where  $C_a^b = \frac{b!}{a!(b-a)!}$ . What we have seen is that only the resonance inducing term is important for the flow and hence the *relevant* part of  $x_0^{2n+1}$  is the  $k = n$  term on the right-hand side of equation (16), namely  $\frac{a^{2n+1}}{2^{2n}} C_n^{2n+1} \cos(t + \theta)$ . From the formula

$$\cos^{2n} \phi = \frac{1}{2^{2n}} C_n^{2n} + \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} C_k^{2n} \cos(2n - 2k)\phi \tag{17}$$

we clearly see that there is no such *relevant* part coming from the  $x_0^{2n+2}$  term of equation (14). Thus, equations (11) and (12), respectively, yield

$$\ddot{x}_{1\epsilon} + x_{1\epsilon} = \sum_{n=0}^{\infty} \frac{b_n}{2n+1} \frac{a^{2n+1}}{2^{2n}} C_n^{2n+1} \sin(t + \theta) + \text{nonresonant terms} \tag{18}$$

$$\ddot{x}_{1\alpha} + x_{1\alpha} = -\sum_{n=1}^{\infty} d_n \frac{a^{2n+1}}{2^{2n}} C_n^{2n+1} \cos(t + \theta) + \text{nonresonant terms}. \tag{19}$$

It is useful to recall that  $\cos(t + \theta)$  or  $\sin(t + \theta)$  on the right-hand side will, respectively, give  $(1/2)t \sin(t + \theta)$  or  $-(1/2)t \cos(t + \theta)$  as the secular term. Thus, including only the divergent part of the solution, to the first non-trivial order,

$$\begin{aligned}
 x &= a \cos(t + \theta) - \frac{\epsilon}{2} \sum_{n=0}^{\infty} \frac{b_n}{2n+1} \frac{a^{2n+1}}{2^{2n}} C_n^{2n+1} t \cos(t + \theta) - \frac{\alpha}{2} \sum_{n=1}^{\infty} d_n \frac{a^{2n+1}}{2^{2n}} C_n^{2n+1} t \sin(t + \theta) \\
 &= a \cos(t + \theta) - \epsilon \sum_{n=0}^{\infty} \frac{b_n}{2n+1} \left(\frac{a}{2}\right)^{2n+1} C_n^{2n+1} (t - \mu + \mu) \cos(t + \theta) \\
 &\quad - \alpha \sum_{n=1}^{\infty} d_n \left(\frac{a}{2}\right)^{2n+1} C_n^{2n+1} (t - \mu + \mu) \sin(t + \theta).
 \end{aligned} \tag{20}$$

The first-order RG equations are obtained by first removing the  $\mu$ -divergence and then imposing

$$\begin{aligned} \frac{dx}{d\mu} &= \frac{da}{d\mu} \cos(t + \theta(\mu)) - a \frac{d\theta}{d\mu} \sin(t + \theta(\mu)) \\ &+ \epsilon \sum_{n=0}^{\infty} \frac{b_n}{2n+1} \left(\frac{a}{2}\right)^{2n+1} C_n^{2n+1} \cos(t + \theta(\mu)) + \alpha \sum_{n=1}^{\infty} d_n \left(\frac{a}{2}\right)^{2n+1} C_n^{2n+1} \sin(t + \theta(\mu)) \\ &= 0. \end{aligned} \tag{21}$$

The amplitude and phase equations follow directly (by comparing coefficients of  $\cos(t + \theta(\mu))$  and  $\sin(t + \theta(\mu))$ ):

$$\frac{da}{d\mu} = -\epsilon \sum_{n=0}^{\infty} \frac{b_n}{2n+1} \left(\frac{a}{2}\right)^{2n+1} C_n^{2n+1} \tag{22}$$

$$\frac{d\theta}{d\mu} = \alpha \sum_{n=1}^{\infty} \frac{d_n}{2} \left(\frac{a}{2}\right)^{2n} C_n^{2n+1}. \tag{23}$$

The interesting information that we get is, to the first order, the amplitude equation is taken care of solely by the  $b_n$ -coefficients (which were the coefficients for the *even* series in the expansion of  $f(x)$ ). The amplitude fixed points are obtained by putting  $\frac{da}{d\mu} = 0$  and then solving for the algebraic polynomial in  $a$ . The roots of that polynomial will yield the fixed points for the amplitude. Stability, as usual, follows from the sign of the derivative. The important point is that in the amplitude equation there are no  $c_n$ -coefficients (which were the coefficients for the *odd* series in the expansion of  $f(x)$ ). Thus, an odd  $f(x)$  ( $b_n = 0$  for all  $n$ ) cannot influence the limit cycle property and hence implies the existence of a centre. It is straightforward to see that this statement is valid to all orders.

To understand this we outline the argument at the second order. Using equations (8) and (9) in equation (6), we obtain the second-order equations as

$$\epsilon^2 : \ddot{x}_{2\epsilon} + x_{2\epsilon} = -x_{1\epsilon} f(x_0) - x_0 x_{1\epsilon} f'(x_0) \tag{24}$$

$$\alpha^2 : \ddot{x}_{2\alpha} + x_{2\alpha} = -x_{1\alpha} g'(x_0) \tag{25}$$

$$\epsilon\alpha : \ddot{x}_{\epsilon\alpha} + x_{\epsilon\alpha} = -x_{1\alpha} f(x_0) - x_0 x_{1\epsilon} f'(x_0) - x_{1\epsilon} g'(x_0), \tag{26}$$

where the functions  $f'$  and  $g'$  are derivatives of  $f$  and  $g$ , respectively, with respect to  $x$ . Precisely,

$$f'(x_0) = \sum_{n=0}^{\infty} [2b_n n x_0^{2n-1} + (2n+1)c_n x_0^{2n}] \tag{27}$$

and

$$g'(x_0) = \sum_{n=0}^{\infty} (2n+1)d_n x_0^{2n}. \tag{28}$$

If  $b_n = 0$ , then the functions  $f(x_0)$  and  $f'(x_0)$  have terms of the forms  $\cos(2l+1)(t + \theta)$  and  $\cos 2l(t + \theta)$  respectively ( $l = 0, 1, 2, \dots$ ). Similarly,  $g(x_0)$  and  $g'(x_0)$  are of the types  $\cos(2l+1)(t + \theta)$  and  $\cos 2l(t + \theta)$ , respectively ( $l = 1, 2, \dots$ ). From equations (11) and (14), we get that (for  $b_n = 0$ ) the renormalized  $x_{1\epsilon}$  has terms only of the form  $\sin 2l(t + \theta)$  (with  $l = 1, 2, \dots$ ). Similarly, from equations (12) and (15), renormalized  $x_{1\alpha}$  has terms of

the form  $\cos(2l + 1)(t + \theta)$  (with  $l = 1, 2, \dots$ ). Therefore, on the right-hand side of equation (24), neither of the two terms contribute a  $\sin(t + \theta)$ . Similarly, the term on the right-hand side of equation (25) does not lead to any  $\sin(t + \theta)$ . On the right-hand side of equation (26), the first term, namely  $x_{1\alpha} f(x_0)$ , has terms of the form  $\sin(2l + 1)(t + \theta) \cos(2k + 1)(t + \theta)$  and hence does not yield a  $\sin(t + \theta)$  (only even multiples occur). Similarly, the term  $x_{1\epsilon} f'(x_0)$  has terms  $\cos(2l + 1)(t + \theta) \cos 2k(t + \theta)$  and hence, again, no  $\sin(t + \theta)$ . Finally, the last term of equation (26), being of the form  $\sin 2l(t + \theta) \cos 2k(t + \theta)$ , also does not yield any  $\sin(t + \theta)$ . Consequently, the expansion for  $x(t)$  has no secular term of the form  $t \cos(t + \theta)$  at  $O(\epsilon^2, \alpha^2, \epsilon\alpha)$ . As a result,  $\frac{da}{d\mu} = 0$  at this order as well. This is the argument that can be carried out systematically to all orders to show that  $\frac{da}{d\mu} = 0$ . The flow  $\frac{d\theta}{d\mu}$  gives correction to the frequency, and for sufficiently small  $\epsilon, \alpha$  and amplitude, we will always have a non-zero value of the frequency, so that an oscillatory state will be achieved with a clear upper limit on the amplitude of the oscillations when  $\alpha = 0$ . One important point needs mention here. In the above analysis, though we saw that no  $\sin(t + \theta)$  term occurs on the right-hand sides of equations (24)–(26), there do occur some  $\cos(t + \theta)$  terms, as can be easily seen. Precisely, they come from the first terms on the right-hand sides of equations (24)–(26). These secular  $\cos(t + \theta)$  terms make contributions to the second-order RG equation for *phase*, i.e.  $\frac{d\theta}{d\mu}$ . Those  $\cos(t + \theta)$  terms which come from the function  $f(x_0)$  obviously contain the  $c_n$ -coefficients. We saw that in equation (23), there were no  $c_n$ -coefficients at the first order. Therefore, an important conclusion is that, if the function  $f(x)$  is odd (i.e.  $b_n = 0$  for all  $n$ ), then the correction to the frequency coming from these  $c_n$ -coefficients does not occur before the second order. The first-order equations are blind to the fact that  $f(x)$  is odd.

This type of RG is also capable of dealing with a Lienard system of the second kind which is not a frequently addressed topic of mathematical physics. They are of the form

$$\ddot{x} + \epsilon f(x)\dot{x}^2 + x + \alpha g(x) = 0. \tag{29}$$

This equation, when compared with equation (6), shows a change in symmetry for the  $f(x)$  term. To study how the behaviour of this system is influenced by the type of the function  $f(x)$ , we go to the simple case of  $\alpha = 0$  in equation (9). The equations for various orders can be found as before,

$$\ddot{x}_0 + x_0 = 0 \tag{30}$$

$$x_{1\epsilon} + x_{1\epsilon} = -\dot{x}_0^2 f(x_0) \tag{31}$$

$$x_{2\epsilon} + x_{2\epsilon} = -\dot{x}_0^2 x_{1\epsilon} f'(x_0) - 2\dot{x}_0 x_{1\epsilon} f(x_0). \tag{32}$$

The solution for  $x_0$  is  $a \cos(t + \theta)$  and the driving term for  $x_1$  is  $-\frac{a^2}{2} (1 - \cos 2(t + \theta)) \sum (b_n x_0^{2n} + c_n x_0^{2n+1})$ . The term  $b_n x_0^{2n}$  can only produce driving terms of the form  $\cos 2l(t + \theta)$  (with  $l = 0, 1, 2, \dots$ ). The resonant driving term comes from  $\frac{a^2}{2} (1 - \cos 2(t + \theta)) \sum c_n x_0^{2n+1}$ . The *relevant* part (i.e. that which leads to a  $\cos(t + \theta)$  term here), according to our earlier discussion (for a fixed  $n$ ), is

$$\begin{aligned} & \frac{a^2}{2} (1 - \cos 2(t + \theta)) \frac{c_n a^{2n+1}}{2^{2n}} \times [C_n^{2n+1} \cos(t + \theta) + C_{n-1}^{2n+1} \cos 3(t + \theta) + \dots] \\ & \Rightarrow c_n \left(\frac{a}{2}\right)^{2n+1} \frac{a^2}{n+2} C_n^{2n+1} \cos(t + \theta) + \text{nonresonant terms.} \end{aligned}$$

Thus, equation (31) takes the form

$$x_{1\epsilon} + x_{1\epsilon} = -c_n \left(\frac{a}{2}\right)^{2n+1} \frac{a^2}{n+2} C_n^{2n+1} \cos(t + \theta) \tag{33}$$

leading to

$$x = a \cos(t + \theta) - \epsilon \frac{a^2}{2} \left[ \sum_{n=0}^{\infty} \left( \frac{a}{2} \right)^{2n+1} \frac{c_n}{n+2} C_n^{2n+1} \right] t \sin(t + \theta) + \text{nondivergent terms.} \quad (34)$$

As before we set  $t = t - \mu + \mu$ , remove the  $\mu$ -divergence and then demand  $\frac{dx}{d\mu} = 0$  to get  $\frac{da}{d\mu} = 0$ , (since there is no  $t \cos(t + \theta)$  term in equation (34)). Finally, from the phase equation we have  $x = a \cos(\Omega t + \theta)$ , with the corrected frequency  $\Omega$  given as

$$\Omega = 1 + \epsilon \sum_{n=0}^{\infty} \left( \frac{a}{2} \right)^{2n+2} \frac{c_n}{n+2} C_n^{2n+1} + \dots \quad (35)$$

The absence of any  $b_n$ -coefficients in this first-order frequency correction shows that the even part of  $f(x)$  has no bearing with this result. An odd  $f(x)$  still supports a centre at the origin, but the amplitude is restricted more strongly than in a Lienard system of the first kind. As opposed to a Lienard system of the first kind, here we see that the  $c_n$ -coefficients enter the frequency correction in the very first order. As an example, our results show that for the Lienard system of the first kind  $\ddot{x} + \epsilon \dot{x}x + x = 0$ , the origin is a centre with the corrected frequency to the lowest order given by  $\Omega = 1 + \frac{\epsilon^2 a}{12} + \dots$ . For the similar Lienard system of the second kind  $\ddot{x} + \epsilon \dot{x}^2 x + x = 0$ , the origin is a centre with the frequency given by  $\Omega = 1 + \frac{\epsilon a^2}{8} + \dots$ .

What if  $f(x)$  were even (i.e.  $c_n = 0$  for all  $n$ ) in equation (29)? In that case, equation (31) yields  $x_1 = \sum_{n,l=0}^{\infty} A_{n,l} \cos 2l(t + \theta)$ . Of the two driving terms in equation (32),  $x_0^2 x_1 \epsilon f'(x_0)$  has the structure  $\cos 2l(t + \theta) \cos(2k + 1)(t + \theta)$ , which means that there will be a  $\cos(t + \theta)$  term but no  $\sin(t + \theta)$ . Similarly, the other term  $x_0 x_1 \epsilon f(x_0)$  has the structure  $\sin(t + \theta) \sin 2l(t + \theta) \cos 2k(t + \theta)$ , which is a combination of odd multiples of cosine and hence admits a  $\cos(t + \theta)$  but again no  $\sin(t + \theta)$ . Thus at  $O(\epsilon^2)$ , we still have  $\frac{da}{d\mu} = 0$ . This argument can be systematically extended and we conclude that limit cycle oscillations are not possible for the Lienard system  $\ddot{x} + \epsilon f(x) \dot{x}^2 + x = 0$ . Inclusion of an additional trapping term,  $\alpha g(x)$  ( $g(x)$  odd function), as written in equation (29), clearly does not change the scenario. The first- and second-order equations for  $x_{1\alpha}$  remain the same as in equations (12) and (25) respectively for the Lienard system of first kind. Therefore, as for the influence of the function  $f(x)$  on the dynamics, the conclusions are clear and strong. For a Lienard system of the first kind, if  $f(x)$  is even, then there are limit cycle oscillations. When  $f(x)$  is an odd function, it supports a centre and the frequency correction does not occur before the second order. For a Lienard system of the second kind, there is no limit cycle oscillation whatsoever. Corrections to frequency come from the odd part of  $f(x)$  at the very first order.

To summarize, starting from a sufficiently general form of the Lienard system of differential equation, we show that a renormalization scheme is capable of bringing the study of the existence of limit cycles and centres on a common platform. This establishes the fact that one need not have to contrive different geometric pictures in different contexts, as is commonly done in the literature, to probe whether a certain Lienard system will have limit cycles or centre. As the study is perturbative, one can also clearly see how the frequency gets corrected in different orders. The approach is general and is capable of addressing a huge class of dynamical systems encountered in practice, as it has been shown here by addressing the two different classes of Lienard systems.

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